# Végeselem analízis

## 3. előadás

Pere Balázs

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2011. október 2.

# ALAPFOGALMAK

# Alapfogalmak

## Kinematikailag lehetséges elmozdulásmező

Egy elmozdulásmezőt kinematikailag lehetségesnek nevezünk, ha

- folytonos függvény,
- elegendően sokszor deriválható a hely szerint a test $V$ térfogatán,
- kielégíti a kinematikai peremfeltételeket a test $A_u$ felületén.

# Alapfogalmak

## Kinematikailag lehetséges elmozdulásmező

Jele: $$\vec{u}^* = \vec{u}^*(\vec{r}) = \vec{u}^*(x, y, z)$$

## Peremfeltételek (kinematikai)

$$\vec{u}^* = \vec{u}_0 \qquad \vec{r} \in A_u$$

## Kinematikailag lehetséges alakváltozás

$$\underline{\underline{A}}^* = \frac{1}{2}\left(\vec{u}^* \circ \nabla + \nabla \circ \vec{u}^*\right)$$

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## Kinematikailag lehetséges feszültség

$$\underline{\underline{F}}^* = \frac{E}{1+\nu}\left(\underline{\underline{A}}^* + \frac{\nu}{1-2\nu} A_I^* \underline{\underline{I}}\right)$$

Általában:

$$\underline{\underline{F}}^* \cdot \nabla + \vec{f} \neq \vec{0}$$

és

$$\underline{\underline{F}}^* \cdot \vec{n} \neq \vec{p}_0.$$

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## Statikailag lehetséges feszültségmező

Egy feszültségmezőt statikailag lehetségesnek nevezünk, ha

- kielégíti az egyensúlyi egyenleteket a test $V$ térfogatán
- kielégíti a dinamikai peremfeltételeket a test $A_p$ felületén

# Alapfogalmak

## Statikailag lehetséges feszültségmező

Jele: $$\underline{\underline{\bar{F}}} = \underline{\underline{\bar{F}}}\left(\vec{r}\right) = \underline{\underline{\bar{F}}}\left(x, y, z\right)$$

## Peremfeltételek (dinamikai)

$$\underline{\underline{\bar{F}}} \cdot \vec{n} = \vec{p}_0$$

## Az egyensúlyi egyenlet

$$\underline{\underline{\bar{F}}} \cdot \nabla + \vec{q} = \vec{0}$$

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$$\underline{\underline{\bar{A}}} = \frac{1+\nu}{E}\left(\underline{\underline{\bar{F}}} - \frac{\nu}{1+\nu}\bar{F}_I \underline{\underline{I}}\right)$$

Általában:

$$\nabla \times \underline{\underline{\bar{A}}} \times \nabla \neq \underline{\underline{0}}$$

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## Virtuális elmozdulásmező

Legyen $\vec{u}_1^*$ és $\vec{u}_2^*$ két kinematikailag lehetséges elmozdulásmező. Virtuális elmozdulásmezőnek nevezzük a

$$\delta\vec{u} := \vec{u}_1^* - \vec{u}_2^*$$

különbséggel definiált függvényt.

## Tulajdonságai

- folytonos függvény,
- elegendően sokszor deriválható,
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## Elmozdulásmező variációja

Legyen $\vec{u}$ a rugalmasságtani feladat egzakt megoldása és $\vec{u}^*$ egy kinematikailag lehetséges elmozdulásmező. Az elmozdulásmező variációjának nevezzük a

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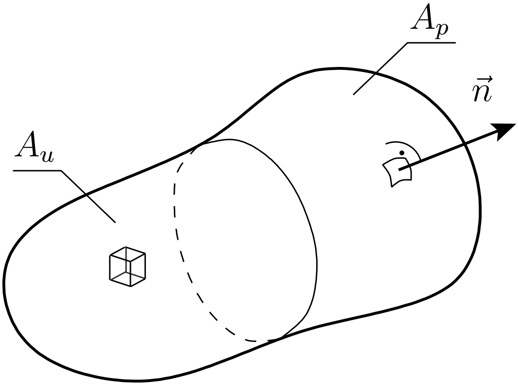
# A RUGALMASSÁGTAN ENERGIA ELVEI

# Virtuális munka elve



# Virtuális munka elve

$$\underline{\underline{\bar{F}}} \cdot \nabla + \vec{f} = \vec{0}$$

$$\downarrow$$

$$\int_{(V)} \left( \vec{u}^* \cdot \underline{\underline{\bar{F}}} \cdot \nabla + \vec{u}^* \cdot \vec{f} \right) dV = 0$$

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## Átalakítás (szorzat deriválási szabálya)

$$\overset{\downarrow}{\overline{\vec{u}^* \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla =$$

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$$\vec{u}^* \cdot \overset{\downarrow}{\underline{\underline{\bar{F}}}} \cdot \nabla = \overset{\downarrow}{\overline{\vec{u}^* \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla - \overset{\downarrow}{\vec{u}^*} \cdot \underline{\underline{\bar{F}}} \cdot \nabla$$

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$$\overset{\downarrow}{\vec{u}^*} \cdot \underline{\underline{\bar{F}}} \cdot \nabla =$$

$$= \overset{\downarrow}{\vec{u}^*} \cdot \left(\bar{\vec{\rho}}_x \circ \vec{e}_x + \bar{\vec{\rho}}_y \circ \vec{e}_y + \bar{\vec{\rho}}_z \circ \vec{e}_z\right) \cdot \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z\right) =$$

$$= \frac{\partial \vec{u}^*}{\partial x} \cdot \left(\bar{\vec{\rho}}_x \circ \vec{e}_x + \bar{\vec{\rho}}_y \circ \vec{e}_y + \bar{\vec{\rho}}_z \circ \vec{e}_z\right) \cdot \vec{e}_x +$$

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# Matematikai kitérő

## Tenzorok kétszeres skaláris szorzata

$$\underline{\underline{A}} \cdot \cdot \underline{\underline{B}} := \mathrm{tr}\left(\underline{\underline{A}}^T \cdot \underline{\underline{B}}\right)$$

ahol $\mathrm{tr}\,(\ldots)$ a tenzor főátlójában lévő elemeinek összegét adja. Pl.

$$\mathrm{tr}\left(\underline{\underline{A}}\right) = A_I$$

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## Következmények

$$\left(\vec{a}\circ\vec{b}\right)\cdot\cdot\left(\vec{c}\circ\vec{d}\right)=$$

$$=\operatorname{tr}\left(\left(\vec{b}\circ\vec{a}\right)\cdot\left(\vec{c}\circ\vec{d}\right)\right)=$$

$$=\operatorname{tr}\left(\vec{b}\circ(\vec{a}\cdot\vec{c})\circ\vec{d}\right)=$$

$$=(\vec{a}\cdot\vec{c})\operatorname{tr}\left(\vec{b}\circ\vec{d}\right)=$$

$$=(\vec{a}\cdot\vec{c})\left(\vec{b}\cdot\vec{d}\right)$$

# Matematikai kitérő

## Következmények

$$\left(\vec{a}\circ\vec{b}\right)\cdot\cdot\left(\vec{c}\circ\vec{d}\right)=$$

$$=\operatorname{tr}\left(\left(\vec{b}\circ\vec{a}\right)\cdot\left(\vec{c}\circ\vec{d}\right)\right)=$$

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$$=(\vec{a}\cdot\vec{c})\left(\vec{b}\cdot\vec{d}\right)$$

# Matematikai kitérő

## Következmények

$$\left(\vec{a} \circ \vec{b}\right) \cdot\cdot \left(\vec{c} \circ \vec{d}\right) =$$

$$= \mathrm{tr}\left(\left(\vec{b} \circ \vec{a}\right) \cdot \left(\vec{c} \circ \vec{d}\right)\right) =$$

$$= \mathrm{tr}\left(\vec{b} \circ (\vec{a} \cdot \vec{c}) \circ \vec{d}\right) =$$

$$= (\vec{a} \cdot \vec{c})\, \mathrm{tr}\left(\vec{b} \circ \vec{d}\right) =$$

$$= (\vec{a} \cdot \vec{c}) \left(\vec{b} \cdot \vec{d}\right)$$

# Matematikai kitérő

## Következmények

$$\left(\vec{a} \circ \vec{b}\right) \cdot\cdot \left(\vec{c} \circ \vec{d}\right) =$$

$$= \mathrm{tr}\left(\left(\vec{b} \circ \vec{a}\right) \cdot \left(\vec{c} \circ \vec{d}\right)\right) =$$

$$= \mathrm{tr}\left(\vec{b} \circ (\vec{a} \cdot \vec{c}) \circ \vec{d}\right) =$$

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# Matematikai kitérő

## Következmények

$$\left(\vec{a} \circ \vec{b}\right) \cdot\cdot \left(\vec{c} \circ \vec{d}\right) =$$

$$= \operatorname{tr}\left(\left(\vec{b} \circ \vec{a}\right) \cdot \left(\vec{c} \circ \vec{d}\right)\right) =$$

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$$= (\vec{a} \cdot \vec{c}) \operatorname{tr}\left(\vec{b} \circ \vec{d}\right) =$$

$$= (\vec{a} \cdot \vec{c}) \left(\vec{b} \cdot \vec{d}\right)$$

# Virtuális munka elve

$$\frac{\partial \vec{u}^*}{\partial x} \cdot \vec{\bar{\rho}}_x + \frac{\partial \vec{u}^*}{\partial y} \cdot \vec{\bar{\rho}}_y + \frac{\partial \vec{u}^*}{\partial z} \cdot \vec{\bar{\rho}}_z =$$

$$= \left( \frac{\partial \vec{u}^*}{\partial x} \cdot \vec{\bar{\rho}}_x \right) (\vec{e}_x \cdot \vec{e}_x) + \left( \frac{\partial \vec{u}^*}{\partial y} \cdot \vec{\bar{\rho}}_y \right) (\vec{e}_y \cdot \vec{e}_y) + \left( \frac{\partial \vec{u}^*}{\partial z} \cdot \vec{\bar{\rho}}_z \right) (\vec{e}_z \cdot \vec{e}_z)$$

# Virtuális munka elve

$$\frac{\partial \vec{u}^*}{\partial x} \cdot \vec{\bar{\rho}}_x + \frac{\partial \vec{u}^*}{\partial y} \cdot \vec{\bar{\rho}}_y + \frac{\partial \vec{u}^*}{\partial z} \cdot \vec{\bar{\rho}}_z =$$

$$= \left( \frac{\partial \vec{u}^*}{\partial x} \cdot \vec{\bar{\rho}}_x \right) (\vec{e}_x \cdot \vec{e}_x) + \left( \frac{\partial \vec{u}^*}{\partial y} \cdot \vec{\bar{\rho}}_y \right) (\vec{e}_y \cdot \vec{e}_y) + \left( \frac{\partial \vec{u}^*}{\partial z} \cdot \vec{\bar{\rho}}_z \right) (\vec{e}_z \cdot \vec{e}_z)$$

# Virtuális munka elve

$$\left(\frac{\partial \vec{u}^*}{\partial x} \cdot \bar{\vec{\rho}}_x\right)(\vec{e}_x \cdot \vec{e}_x) + \left(\frac{\partial \vec{u}^*}{\partial y} \cdot \bar{\vec{\rho}}_y\right)(\vec{e}_y \cdot \vec{e}_y) + \left(\frac{\partial \vec{u}^*}{\partial z} \cdot \bar{\vec{\rho}}_z\right)(\vec{e}_z \cdot \vec{e}_z) =$$

$$= \left(\frac{\partial \vec{u}^*}{\partial x} \cdot \bar{\vec{\rho}}_x\right)(\vec{e}_x \cdot \vec{e}_x) + \left(\frac{\partial \vec{u}^*}{\partial y} \cdot \bar{\vec{\rho}}_y\right)(\vec{e}_y \cdot \vec{e}_y) + \left(\frac{\partial \vec{u}^*}{\partial z} \cdot \bar{\vec{\rho}}_z\right)(\vec{e}_z \cdot \vec{e}_z) +$$

$$+ \left(\frac{\partial \vec{u}^*}{\partial y} \cdot \bar{\vec{\rho}}_x\right)(\vec{e}_y \cdot \vec{e}_x) + \left(\frac{\partial \vec{u}^*}{\partial z} \cdot \bar{\vec{\rho}}_y\right)(\vec{e}_z \cdot \vec{e}_y) + \left(\frac{\partial \vec{u}^*}{\partial x} \cdot \bar{\vec{\rho}}_z\right)(\vec{e}_x \cdot \vec{e}_z) +$$

$$+ \left(\frac{\partial \vec{u}^*}{\partial z} \cdot \bar{\vec{\rho}}_x\right)(\vec{e}_z \cdot \vec{e}_x) + \left(\frac{\partial \vec{u}^*}{\partial x} \cdot \bar{\vec{\rho}}_y\right)(\vec{e}_x \cdot \vec{e}_y) + \left(\frac{\partial \vec{u}^*}{\partial y} \cdot \bar{\vec{\rho}}_z\right)(\vec{e}_y \cdot \vec{e}_z) =$$

# Virtuális munka elve

$$\left(\frac{\partial \vec{u}^*}{\partial x}\cdot\bar{\vec{\rho}}_x\right)(\vec{e}_x\cdot\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial y}\cdot\bar{\vec{\rho}}_y\right)(\vec{e}_y\cdot\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial z}\cdot\bar{\vec{\rho}}_z\right)(\vec{e}_z\cdot\vec{e}_z)=$$

$$=\left(\frac{\partial \vec{u}^*}{\partial x}\cdot\bar{\vec{\rho}}_x\right)(\vec{e}_x\cdot\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial y}\cdot\bar{\vec{\rho}}_y\right)(\vec{e}_y\cdot\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial z}\cdot\bar{\vec{\rho}}_z\right)(\vec{e}_z\cdot\vec{e}_z)+$$

$$+\left(\frac{\partial \vec{u}^*}{\partial y}\cdot\bar{\vec{\rho}}_x\right)(\vec{e}_y\cdot\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial z}\cdot\bar{\vec{\rho}}_y\right)(\vec{e}_z\cdot\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial x}\cdot\bar{\vec{\rho}}_z\right)(\vec{e}_x\cdot\vec{e}_z)+$$

$$+\left(\frac{\partial \vec{u}^*}{\partial z}\cdot\bar{\vec{\rho}}_x\right)(\vec{e}_z\cdot\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial x}\cdot\bar{\vec{\rho}}_y\right)(\vec{e}_x\cdot\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial y}\cdot\bar{\vec{\rho}}_z\right)(\vec{e}_y\cdot\vec{e}_z)=$$

# Virtuális munka elve

$$\left(\frac{\partial \vec{u}^*}{\partial x}\cdot\bar{\vec{\rho}}_x\right)(\vec{e}_x\cdot\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial y}\cdot\bar{\vec{\rho}}_y\right)(\vec{e}_y\cdot\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial z}\cdot\bar{\vec{\rho}}_z\right)(\vec{e}_z\cdot\vec{e}_z)=$$

$$=\left(\frac{\partial \vec{u}^*}{\partial x}\circ\vec{e}_x\right)\cdot\cdot\,(\bar{\vec{\rho}}_x\circ\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial y}\circ\vec{e}_y\right)\cdot\cdot\,(\bar{\vec{\rho}}_y\circ\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial z}\circ\vec{e}_z\right)\cdot\cdot\,(\bar{\vec{\rho}}_z\circ\vec{e}_z)+$$

$$=\left(\frac{\partial \vec{u}^*}{\partial y}\circ\vec{e}_y\right)\cdot\cdot\,(\bar{\vec{\rho}}_x\circ\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial z}\circ\vec{e}_z\right)\cdot\cdot\,(\bar{\vec{\rho}}_y\circ\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial x}\circ\vec{e}_x\right)\cdot\cdot\,(\bar{\vec{\rho}}_z\circ\vec{e}_z)+$$

$$=\left(\frac{\partial \vec{u}^*}{\partial z}\circ\vec{e}_z\right)\cdot\cdot\,(\bar{\vec{\rho}}_x\circ\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial x}\circ\vec{e}_x\right)\cdot\cdot\,(\bar{\vec{\rho}}_y\circ\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial y}\circ\vec{e}_y\right)\cdot\cdot\,(\bar{\vec{\rho}}_z\circ\vec{e}_z)+$$

$$=\left(\frac{\partial \vec{u}^*}{\partial x}\circ\vec{e}_x+\frac{\partial \vec{u}^*}{\partial y}\circ\vec{e}_y+\frac{\partial \vec{u}^*}{\partial z}\circ\vec{e}_z\right)\cdot\cdot\,(\bar{\vec{\rho}}_x\circ\vec{e}_x+\bar{\vec{\rho}}_y\circ\vec{e}_y+\bar{\vec{\rho}}_z\circ\vec{e}_z)=$$

# Virtuális munka elve

$$\left(\frac{\partial \vec{u}^*}{\partial x}\cdot\bar{\vec{\rho}}_x\right)(\vec{e}_x\cdot\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial y}\cdot\bar{\vec{\rho}}_y\right)(\vec{e}_y\cdot\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial z}\cdot\bar{\vec{\rho}}_z\right)(\vec{e}_z\cdot\vec{e}_z)=$$

$$=\left(\frac{\partial \vec{u}^*}{\partial x}\circ\vec{e}_x\right)\cdot\cdot\,(\bar{\vec{\rho}}_x\circ\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial y}\circ\vec{e}_y\right)\cdot\cdot\,(\bar{\vec{\rho}}_y\circ\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial z}\circ\vec{e}_z\right)\cdot\cdot\,(\bar{\vec{\rho}}_z\circ\vec{e}_z)+$$

$$=\left(\frac{\partial \vec{u}^*}{\partial y}\circ\vec{e}_y\right)\cdot\cdot\,(\bar{\vec{\rho}}_x\circ\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial z}\circ\vec{e}_z\right)\cdot\cdot\,(\bar{\vec{\rho}}_y\circ\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial x}\circ\vec{e}_x\right)\cdot\cdot\,(\bar{\vec{\rho}}_z\circ\vec{e}_z)+$$

$$=\left(\frac{\partial \vec{u}^*}{\partial z}\circ\vec{e}_z\right)\cdot\cdot\,(\bar{\vec{\rho}}_x\circ\vec{e}_x)+\left(\frac{\partial \vec{u}^*}{\partial x}\circ\vec{e}_x\right)\cdot\cdot\,(\bar{\vec{\rho}}_y\circ\vec{e}_y)+\left(\frac{\partial \vec{u}^*}{\partial y}\circ\vec{e}_y\right)\cdot\cdot\,(\bar{\vec{\rho}}_z\circ\vec{e}_z)+$$

$$=\left(\frac{\partial \vec{u}^*}{\partial x}\circ\vec{e}_x+\frac{\partial \vec{u}^*}{\partial y}\circ\vec{e}_y+\frac{\partial \vec{u}^*}{\partial z}\circ\vec{e}_z\right)\cdot\cdot\,(\bar{\vec{\rho}}_x\circ\vec{e}_x+\bar{\vec{\rho}}_y\circ\vec{e}_y+\bar{\vec{\rho}}_z\circ\vec{e}_z)=$$

# Virtuális munka elve

$$\left(\frac{\partial \vec{u}^*}{\partial x}\circ\vec{e}_x+\frac{\partial \vec{u}^*}{\partial y}\circ\vec{e}_y+\frac{\partial \vec{u}^*}{\partial z}\circ\vec{e}_z\right)\cdot\cdot\left(\bar{\vec{\rho}}_x\circ\vec{e}_x+\bar{\vec{\rho}}_y\circ\vec{e}_y+\bar{\vec{\rho}}_z\circ\vec{e}_z\right)=$$

$$=\underline{\underline{D}}^*\cdot\cdot\underline{\underline{\bar{F}}}=\underline{\underline{\bar{F}}}\cdot\cdot\underline{\underline{D}}^*$$

# Virtuális munka elve

$$\underline{\underline{D}}^* = \underline{\underline{A}}^* + \underline{\underline{\Psi}}^*$$

$$\Downarrow$$

$$\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{D}}^* = \underline{\underline{\bar{F}}} \cdot \cdot \left(\underline{\underline{A}}^* + \underline{\underline{\Psi}}^*\right) = \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* + \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^*$$

# Virtuális munka elve

$$\underline{\underline{D}}^* = \underline{\underline{A}}^* + \underline{\underline{\Psi}}^*$$

$$\Downarrow$$

$$\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{D}}^* = \underline{\underline{\bar{F}}} \cdot \cdot \left( \underline{\underline{A}}^* + \underline{\underline{\Psi}}^* \right) = \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* + \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^*$$

# Virtuális munka elve

$$\underline{\underline{\bar{F}}} \cdot\cdot \underline{\underline{\Psi}}^{*} = \underline{\underline{\bar{F}}}^{T} \cdot\cdot \left(\underline{\underline{\Psi}}^{*}\right)^{T} = -\underline{\underline{\bar{F}}} \cdot\cdot \underline{\underline{\Psi}}^{*}$$

# Virtuális munka elve

$$\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^* = \underline{\underline{\bar{F}}}^T \cdot \cdot \left(\underline{\underline{\Psi}}^*\right)^T = -\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^*$$

# Virtuális munka elve

$$\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^* = \underline{\underline{\bar{F}}}^T \cdot \cdot \left(\underline{\underline{\Psi}}^*\right)^T = -\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^*$$

# Virtuális munka elve

$$\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^* = \underline{\underline{\bar{F}}}^T \cdot \cdot \left(\underline{\underline{\Psi}}^*\right)^T = -\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^*$$

$$\Downarrow$$

$$\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^* = 0$$

# Virtuális munka elve

$$\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^* = \underline{\underline{\bar{F}}}^T \cdot \cdot \left(\underline{\underline{\Psi}}^*\right)^T = -\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^*$$

$$\Downarrow$$

$$\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{\Psi}}^* = 0$$

$$\Downarrow$$

$$\underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{D}}^* = \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^*$$

# Virtuális munka elve

$$\int_{(V)} \left( \overset{\downarrow}{\overline{\vec{u}^* \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla - \overset{\downarrow}{\vec{u}^*} \cdot \underline{\underline{\bar{F}}} \cdot \nabla + \vec{u}^* \cdot \vec{f} \right) dV = 0$$

$$\Downarrow$$

$$\int_{(V)} \left( \overset{\downarrow}{\overline{\vec{u}^* \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla - \underline{\underline{\bar{F}}} \cdot\cdot \underline{\underline{A}}^* + \vec{u}^* \cdot \vec{f} \right) dV = 0$$

$$\Downarrow$$

$$\int_{(V)} \overset{\downarrow}{\overline{\vec{u}^* \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla dV - \int_{(V)} \underline{\underline{\bar{F}}} \cdot\cdot \underline{\underline{A}}^* dV + \int_{(V)} \vec{u}^* \cdot \vec{f} dV = 0$$

# Virtuális munka elve

$$\int\limits_{(V)} \left( \overline{\overset{\downarrow}{\vec{u}^{*}} \cdot \underline{\underline{\bar{F}}}} \cdot \nabla - \overset{\downarrow}{\vec{u}^{*}} \cdot \underline{\underline{\bar{F}}} \cdot \nabla + \vec{u}^{*} \cdot \vec{f} \right) dV = 0$$

$$\Downarrow$$

$$\int\limits_{(V)} \left( \overline{\overset{\downarrow}{\vec{u}^{*} \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla - \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^{*} + \vec{u}^{*} \cdot \vec{f} \right) dV = 0$$

$$\Downarrow$$

$$\int\limits_{(V)} \overline{\overset{\downarrow}{\vec{u}^{*} \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla dV - \int\limits_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^{*} dV + \int\limits_{(V)} \vec{u}^{*} \cdot \vec{f} dV = 0$$

# Virtuális munka elve

$$\int\limits_{(V)} \left( \overset{\downarrow}{\overline{\vec{u}^* \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla - \overset{\downarrow}{\vec{u}^*} \cdot \underline{\underline{\bar{F}}} \cdot \nabla + \vec{u}^* \cdot \vec{f} \right) dV = 0$$

$$\Downarrow$$

$$\int\limits_{(V)} \left( \overset{\downarrow}{\overline{\vec{u}^* \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla - \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* + \vec{u}^* \cdot \vec{f} \right) dV = 0$$

$$\Downarrow$$

$$\int\limits_{(V)} \overset{\downarrow}{\overline{\vec{u}^* \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla dV - \int\limits_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* dV + \int\limits_{(V)} \vec{u}^* \cdot \vec{f} dV = 0$$

# Virtuális munka elve

## Gauss-tétel

$$\int\limits_{(V)} \overset{\downarrow}{\overline{\vec{u}^* \cdot \underline{\underline{\bar{F}}}}} \cdot \nabla dV - \int\limits_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* dV + \int\limits_{(V)} \vec{u}^* \cdot \vec{f} dV = 0$$

$$\Downarrow$$

$$\int\limits_{(A)} \vec{u}^* \cdot \underline{\underline{\bar{F}}} \cdot \vec{n}\, dA - \int\limits_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* dV + \int\limits_{(V)} \vec{u}^* \cdot \vec{f} dV = 0$$

# Virtuális munka elve

## Gauss-tétel

$$\int_{(V)} \overbrace{\vec{u}^* \cdot \underline{\underline{\bar{F}}}}^{\downarrow} \cdot \nabla dV - \int_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* dV + \int_{(V)} \vec{u}^* \cdot \vec{f} dV = 0$$

$$\Downarrow$$

$$\int_{(A)} \vec{u}^* \cdot \underline{\underline{\bar{F}}} \cdot \vec{n}\, dA - \int_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* dV + \int_{(V)} \vec{u}^* \cdot \vec{f} dV = 0$$

# Virtuális munka elve

## Virtuális munka elve

$$\int_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* dV - \int_{(A)} \vec{u}^* \cdot \underline{\underline{\bar{F}}} \cdot \vec{n}\, dA - \int_{(V)} \vec{u}^* \cdot \vec{f} dV = 0$$

$$\Downarrow$$

$$\int_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* dV - \int_{(A_u)} \vec{u}_0 \cdot \underline{\underline{\bar{F}}} \cdot \vec{n}\, dA - \int_{(A_p)} \vec{u}^* \cdot \vec{p}_0 dA - \int_{(V)} \vec{u}^* \cdot \vec{f} dV = 0$$

# Virtuális munka elve

## Virtuális munka elve

$$\int_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* dV - \int_{(A)} \vec{u}^* \cdot \underline{\underline{\bar{F}}} \cdot \vec{n}\, dA - \int_{(V)} \vec{u}^* \cdot \vec{f} dV = 0$$

$$\Downarrow$$

$$\int_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}^* dV - \int_{(A_u)} \vec{u}_0 \cdot \underline{\underline{\bar{F}}} \cdot \vec{n}\, dA - \int_{(A_p)} \vec{u}^* \cdot \vec{p}_0 dA - \int_{(V)} \vec{u}^* \cdot \vec{f} dV = 0$$

# Virtuális elmozdulás elve

$$\int\limits_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}_1^* dV - \int\limits_{(A_u)} \vec{u}_0 \cdot \underline{\underline{\bar{F}}} \cdot \vec{n} \, dA - \int\limits_{(A_p)} \vec{u}_1^* \cdot \vec{p}_0 dA - \int\limits_{(V)} \vec{u}_1^* \cdot \vec{f} dV = 0$$

$$\int\limits_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \underline{\underline{A}}_2^* dV - \int\limits_{(A_u)} \vec{u}_0 \cdot \underline{\underline{\bar{F}}} \cdot \vec{n} \, dA - \int\limits_{(A_p)} \vec{u}_2^* \cdot \vec{p}_0 dA - \int\limits_{(V)} \vec{u}_2^* \cdot \vec{f} dV = 0$$

# Virtuális elmozdulás elve

$$\int\limits_{(V)} \underline{\underline{\bar{F}}} \cdot\cdot \left( \underline{\underline{A}}_1^* - \underline{\underline{A}}_2^* \right) dV - \underbrace{\int\limits_{(A_u)} (\vec{u}_1^* - \vec{u}_2^*) \cdot \underline{\underline{\bar{F}}} \cdot \vec{n}\, dA}_{0} -$$

$$- \int\limits_{(A_p)} (\vec{u}_1^* - \vec{u}_2^*) \cdot \vec{p}_0 dA - \int\limits_{(V)} (\vec{u}_1^* - \vec{u}_2^*) \cdot \vec{f} dV = 0$$

# Virtuális elmozdulás elve

$$\vec{u}_1^* - \vec{u}_2^* = \delta\vec{u}$$

# Virtuális elmozdulás elve

$$\vec{u}_1^* - \vec{u}_2^* = \delta \vec{u}$$

$$\underline{\underline{A}}_1^* - \underline{\underline{A}}_2^* = \frac{1}{2}\left(\vec{u}_1^* \circ \nabla + \nabla \circ \vec{u}_1^*\right) - \frac{1}{2}\left(\vec{u}_2^* \circ \nabla + \nabla \circ \vec{u}_2^*\right) =$$

# Virtuális elmozdulás elve

$$\vec{u}_1^* - \vec{u}_2^* = \delta\vec{u}$$

$$\underline{\underline{A}}_1^* - \underline{\underline{A}}_2^* = \frac{1}{2}\left(\vec{u}_1^* \circ \nabla + \nabla \circ \vec{u}_1^*\right) - \frac{1}{2}\left(\vec{u}_2^* \circ \nabla + \nabla \circ \vec{u}_2^*\right) =$$

# Virtuális elmozdulás elve

$$\vec{u}_1^* - \vec{u}_2^* = \delta\vec{u}$$

$$\underline{\underline{A}}_1^* - \underline{\underline{A}}_2^* = \frac{1}{2}\left(\vec{u}_1^* \circ \nabla + \nabla \circ \vec{u}_1^*\right) - \frac{1}{2}\left(\vec{u}_2^* \circ \nabla + \nabla \circ \vec{u}_2^*\right) =$$

$$= \frac{1}{2}\left(\left(\vec{u}_1^* - \vec{u}_2^*\right) \circ \nabla + \nabla \circ \left(\vec{u}_1^* - \vec{u}_2^*\right)\right) = \frac{1}{2}\left(\delta\vec{u} \circ \nabla + \nabla \circ \delta\vec{u}\right) = \delta\underline{\underline{A}}$$

# Virtuális elmozdulás elve

$$\vec{u}_1^* - \vec{u}_2^* = \delta\vec{u}$$

$$\underline{\underline{A}}_1^* - \underline{\underline{A}}_2^* = \frac{1}{2}\left(\vec{u}_1^* \circ \nabla + \nabla \circ \vec{u}_1^*\right) - \frac{1}{2}\left(\vec{u}_2^* \circ \nabla + \nabla \circ \vec{u}_2^*\right) =$$

$$= \frac{1}{2}\left(\left(\vec{u}_1^* - \vec{u}_2^*\right) \circ \nabla + \nabla \circ \left(\vec{u}_1^* - \vec{u}_2^*\right)\right) = \frac{1}{2}\left(\delta\vec{u} \circ \nabla + \nabla \circ \delta\vec{u}\right) = \delta\underline{\underline{A}}$$

# Virtuális elmozdulás elve

$$\vec{u}_1^* - \vec{u}_2^* = \delta\vec{u}$$

$$\underline{\underline{A}}_1^* - \underline{\underline{A}}_2^* = \frac{1}{2}\left(\vec{u}_1^* \circ \nabla + \nabla \circ \vec{u}_1^*\right) - \frac{1}{2}\left(\vec{u}_2^* \circ \nabla + \nabla \circ \vec{u}_2^*\right) =$$

$$= \frac{1}{2}\left((\vec{u}_1^* - \vec{u}_2^*) \circ \nabla + \nabla \circ (\vec{u}_1^* - \vec{u}_2^*)\right) = \frac{1}{2}\left(\delta\vec{u} \circ \nabla + \nabla \circ \delta\vec{u}\right) = \delta\underline{\underline{A}}$$

# Virtuális elmozdulás elve

## Virtuális elmozdulás elve...

...vagy a feladat gyenge alakja:

$$\int\limits_{(V)} \underline{\underline{\bar{F}}} \cdot \cdot \delta \underline{\underline{A}} dV - \int\limits_{(A_p)} \delta \vec{u} \cdot \vec{p}_0 dA - \int\limits_{(V)} \delta \vec{u} \cdot \vec{f} dV = 0$$