Végeselem analízis

4. előadás

Pere Balázs

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A RUGALMASSÁGTAN ENERGIA ELVEI

Potenciális energia minimuma elv

Potenciális energia

$$\Pi_p := U - W_k$$

ahol

- U az alakváltozási energia,
- W_k a külső erők (virtuális) munkája.

Potenciális energia minimuma elv

Alakváltozási energia

$$U = \frac{1}{2} \int_{(V)} \underline{\underline{F}} \cdot \cdot \underline{\underline{A}} dV$$

Külső erők (virtuális) munkája

$$W_k = \int_{(A_p)} \vec{u} \cdot \vec{p}_0 \, dA + \int_{(V)} \vec{u} \cdot \vec{f} \, dV$$

Potenciális energia minimuma elv

Alakváltozási energia

$$U = \frac{1}{2} \int\limits_{(V)} \underline{\underline{F}} \cdot \cdot \underline{\underline{A}} dV$$

Külső erők (virtuális) munkája

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Potenciális energia minimuma elv

- $\underline{\underline{F}}$ a feszültségi tenzor,
- $\underline{\underline{A}}$ pedig az alakváltozási tenzor.
- $\vec{p}_0$ az A_p felület pontjaiban az egységnyi felületre jutó terhelés,
- $\vec{f}$ a V térfogaton belül az egységnyi térfogatra jutó terhelés,
- $\vec{u}$ az anyagi pont elmozdulása.

Potenciális energia minimuma elv

A potenciális energia, mint funkcionál

$$\Pi_p = \frac{1}{2} \int\limits_{(V)} \underline{\underline{F}} \cdot \cdot \underline{\underline{A}} dV - \int\limits_{(A_p)} \vec{u} \cdot \vec{p}_0 \, dA - \int\limits_{(V)} \vec{u} \cdot \vec{f} \, dV$$

Potenciális energia minimuma elv

A potenciális energia, mint funkcionál

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$$\underline{\underline{A}} = \frac{1}{2} \left(\vec{u} \circ \nabla + \nabla \circ \vec{u} \right)$$

$$\Downarrow$$

$$\underline{\underline{A}} = \underline{\underline{A}} \left(\vec{u} \right)$$

Potenciális energia minimuma elv

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Potenciális energia minimuma elv

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$$\underline{\underline{F}} = \frac{E}{1+\nu} \left(\underline{\underline{A}} + \frac{\nu}{1-2\nu} A_I \underline{\underline{I}} \right)$$

$$\Downarrow$$

$$\underline{\underline{F}} = \underline{\underline{F}} \left(\underline{\underline{A}} \right) = \underline{\underline{F}} \left(\underline{\underline{A}} \left(\vec{u} \right) \right) = \underline{\underline{F}} \left(\vec{u} \right)$$

Potenciális energia minimuma elv

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Potenciális energia minimuma elv

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$$\Pi_p = \Pi_p \left[\vec{u} \left(\vec{r} \right) \right]$$

Potenciális energia minimuma elv

A potenciális energia minimuma elv

$$\Pi_p\left[\vec{u}^*\right] \geqq \Pi_p\left[\vec{u}\right]$$

Potenciális energia minimuma elv

Bizonyítás

$$\delta\vec{u} = \vec{u}^* - \vec{u}$$

$$\Downarrow$$

$$\vec{u}^* = \vec{u} + \delta\vec{u}$$

Bizonyítás

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Bizonyítás

$$\underline{\underline{A}}^* = \frac{1}{2}\left(\vec{u}^* \circ \nabla + \nabla \circ \vec{u}^*\right) =$$

$$= \frac{1}{2}\left((\vec{u} + \delta\vec{u}) \circ \nabla + \nabla \circ (\vec{u} + \delta\vec{u})\right) =$$

$$= \frac{1}{2}\left(\vec{u} \circ \nabla + \nabla \circ \vec{u}\right) + \frac{1}{2}\left(\delta\vec{u} \circ \nabla + \nabla \circ \delta\vec{u}\right) =$$

$$= \underline{\underline{A}} + \delta\underline{\underline{A}}$$

Potenciális energia minimuma elv

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Potenciális energia minimuma elv

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Potenciális energia minimuma elv

Bizonyítás

$$\underline{\underline{F}}^* = \frac{E}{1+\nu}\left(\underline{\underline{A}}^* + \frac{\nu}{1-2\nu}A_I^*\underline{\underline{I}}\right) = \frac{E}{1+\nu}\left[\underline{\underline{A}}^* + \frac{\nu}{1-2\nu}\left(\underline{\underline{A}}^* \cdot\cdot \underline{\underline{I}}\right)\underline{\underline{I}}\right] =$$

$$= \frac{E}{1+\nu}\left[\underline{\underline{A}} + \delta\underline{\underline{A}} + \frac{\nu}{1-2\nu}\left(\left(\underline{\underline{A}} + \delta\underline{\underline{A}}\right) \cdot\cdot \underline{\underline{I}}\right)\underline{\underline{I}}\right] =$$

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Potenciális energia minimuma elv

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Potenciális energia minimuma elv

Bizonyítás

$$\Pi_p\left[\vec{u}^*\right] = \Pi_p\left[\vec{u} + \delta\vec{u}\right] =$$

$$= \frac{1}{2}\int\limits_{(V)} \underline{\underline{F}}^* \cdot\cdot \underline{\underline{A}}^* dV - \int\limits_{(A_p)} \vec{u}^* \cdot \vec{p}_0\, dA - \int\limits_{(V)} \vec{u}^* \cdot \vec{f}\, dV =$$

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Potenciális energia minimuma elv

Bizonyítás

$$= \frac{1}{2} \int\limits_{(V)} \underline{\underline{F}} \cdot \cdot \underline{\underline{A}} dV - \int\limits_{(A_p)} \vec{u} \cdot \vec{p}_0 \, dA - \int\limits_{(V)} \vec{u} \cdot \vec{f} \, dV +$$

$$+ \frac{1}{2} \int\limits_{(V)} \left(\underline{\underline{F}} \cdot \cdot \delta \underline{\underline{A}} + \delta \underline{\underline{F}} \cdot \cdot \underline{\underline{A}} \right) dV - \int\limits_{(A_p)} \delta \vec{u} \cdot \vec{p}_0 \, dA - \int\limits_{(V)} \delta \vec{u} \cdot \vec{f} \, dV +$$

$$+ \frac{1}{2} \int\limits_{(V)} \delta \underline{\underline{F}} \cdot \cdot \delta \underline{\underline{A}} \, dV$$

Potenciális energia minimuma elv

Bizonyítás

$$= \frac{1}{2}\int\limits_{(V)} \underline{\underline{F}} \cdot \cdot \underline{\underline{A}} dV - \int\limits_{(A_p)} \vec{u} \cdot \vec{p}_0 \, dA - \int\limits_{(V)} \vec{u} \cdot \vec{f} \, dV +$$

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Potenciális energia minimuma elv

Bizonyítás

$$\delta \underline{\underline{F}} \cdot\cdot \underline{\underline{A}} = \frac{E}{1+\nu}\left(\delta \underline{\underline{A}} + \frac{\nu}{1-2\nu}\delta A_I \underline{\underline{I}}\right) \cdot\cdot \underline{\underline{A}} =$$

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Potenciális energia minimuma elv

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$$\Pi_p\left[\vec{u}^*\right] = \underbrace{\frac{1}{2}\int\limits_{(V)} \underline{\underline{F}} \cdot\cdot \underline{\underline{A}} dV - \int\limits_{(A_p)} \vec{u}\cdot\vec{p}_0\,dA - \int\limits_{(V)} \vec{u}\cdot\vec{f}\,dV}_{\Pi_p[\vec{u}]} +$$

$$+ \underbrace{\int\limits_{(V)} \underline{\underline{F}} \cdot\cdot \delta\underline{\underline{A}}\,dV - \int\limits_{(A_p)} \delta\vec{u}\cdot\vec{p}_0\,dA - \int\limits_{(V)} \delta\vec{u}\cdot\vec{f}\,dV}_{\delta\Pi_p=0} +$$

$$+ \underbrace{\frac{1}{2}\int\limits_{(V)} \delta\underline{\underline{F}} \cdot\cdot \delta\underline{\underline{A}}\,dV}_{\delta^2\Pi_p}$$

Potenciális energia minimuma elv

Bizonyítás

$$\delta\underline{\underline{F}}\cdot\cdot\delta\underline{\underline{A}}=\frac{E}{1+\nu}\left(\delta\underline{\underline{A}}+\frac{\nu}{1-2\nu}\delta A_I\underline{\underline{I}}\right)\cdot\cdot\delta\underline{\underline{A}}=$$

$$=\frac{E}{1+\nu}\delta\underline{\underline{A}}\cdot\cdot\delta\underline{\underline{A}}+\frac{E}{1+\nu}\frac{\nu}{1-2\nu}\delta A_I\underbrace{\underline{\underline{I}}\cdot\cdot\delta\underline{\underline{A}}}_{\delta A_I}=$$

$$=\frac{E}{1+\nu}(\delta\vec{\alpha}_x\circ\vec{e}_x+\delta\vec{\alpha}_y\circ\vec{e}_y+\delta\vec{\alpha}_z\circ\vec{e}_z)\cdot\cdot(\delta\vec{\alpha}_x\circ\vec{e}_x+\delta\vec{\alpha}_y\circ\vec{e}_y+\delta\vec{\alpha}_z\circ\vec{e}_z)+$$

$$+\frac{E}{1+\nu}\frac{\nu}{1-2\nu}(\delta A_I)^2=$$

$$=\frac{E}{1+\nu}(\delta\vec{\alpha}_x\cdot\delta\vec{\alpha}_x+\delta\vec{\alpha}_y\cdot\delta\vec{\alpha}_y+\delta\vec{\alpha}_z\cdot\delta\vec{\alpha}_z)+\frac{E}{1+\nu}\frac{\nu}{1-2\nu}(\delta A_I)^2$$

Potenciális energia minimuma elv

Bizonyítás

$$\delta \underline{\underline{F}} \cdot\cdot \delta \underline{\underline{A}} = \frac{E}{1+\nu}\left(\delta \underline{\underline{A}} + \frac{\nu}{1-2\nu}\delta A_I \underline{\underline{I}}\right) \cdot\cdot \delta \underline{\underline{A}} =$$

$$= \frac{E}{1+\nu}\delta \underline{\underline{A}} \cdot\cdot \delta \underline{\underline{A}} + \frac{E}{1+\nu}\frac{\nu}{1-2\nu}\delta A_I \underbrace{\underline{\underline{I}} \cdot\cdot \delta \underline{\underline{A}}}_{\delta A_I} =$$

$$= \frac{E}{1+\nu}(\delta\vec{\alpha}_x \circ \vec{e}_x + \delta\vec{\alpha}_y \circ \vec{e}_y + \delta\vec{\alpha}_z \circ \vec{e}_z) \cdot\cdot (\delta\vec{\alpha}_x \circ \vec{e}_x + \delta\vec{\alpha}_y \circ \vec{e}_y + \delta\vec{\alpha}_z \circ \vec{e}_z) +$$

$$+ \frac{E}{1+\nu}\frac{\nu}{1-2\nu}(\delta A_I)^2 =$$

$$= \frac{E}{1+\nu}(\delta\vec{\alpha}_x \cdot \delta\vec{\alpha}_x + \delta\vec{\alpha}_y \cdot \delta\vec{\alpha}_y + \delta\vec{\alpha}_z \cdot \delta\vec{\alpha}_z) + \frac{E}{1+\nu}\frac{\nu}{1-2\nu}(\delta A_I)^2$$

Potenciális energia minimuma elv

Bizonyítás

$$\delta \underline{\underline{F}} \cdot \cdot \delta \underline{\underline{A}} = \frac{E}{1+\nu}\left(\delta \underline{\underline{A}} + \frac{\nu}{1-2\nu}\delta A_I \underline{\underline{I}}\right) \cdot \cdot \delta \underline{\underline{A}} =$$

$$= \frac{E}{1+\nu}\delta \underline{\underline{A}} \cdot \cdot \delta \underline{\underline{A}} + \frac{E}{1+\nu}\frac{\nu}{1-2\nu}\delta A_I \underbrace{\underline{\underline{I}} \cdot \cdot \delta \underline{\underline{A}}}_{\delta A_I} =$$

$$= \frac{E}{1+\nu}(\delta\vec{\alpha}_x \circ \vec{e}_x + \delta\vec{\alpha}_y \circ \vec{e}_y + \delta\vec{\alpha}_z \circ \vec{e}_z) \cdot \cdot (\delta\vec{\alpha}_x \circ \vec{e}_x + \delta\vec{\alpha}_y \circ \vec{e}_y + \delta\vec{\alpha}_z \circ \vec{e}_z) +$$

$$+ \frac{E}{1+\nu}\frac{\nu}{1-2\nu}(\delta A_I)^2 =$$

$$= \frac{E}{1+\nu}(\delta\vec{\alpha}_x \cdot \delta\vec{\alpha}_x + \delta\vec{\alpha}_y \cdot \delta\vec{\alpha}_y + \delta\vec{\alpha}_z \cdot \delta\vec{\alpha}_z) + \frac{E}{1+\nu}\frac{\nu}{1-2\nu}(\delta A_I)^2$$

Potenciális energia minimuma elv

Bizonyítás

$$\delta \underline{\underline{F}} \cdot \cdot \delta \underline{\underline{A}} = \frac{E}{1+\nu} \left(\delta \underline{\underline{A}} + \frac{\nu}{1-2\nu} \delta A_I \underline{\underline{I}} \right) \cdot \cdot \delta \underline{\underline{A}} =$$

$$= \frac{E}{1+\nu} \delta \underline{\underline{A}} \cdot \cdot \delta \underline{\underline{A}} + \frac{E}{1+\nu} \frac{\nu}{1-2\nu} \delta A_I \underbrace{\underline{\underline{I}} \cdot \cdot \delta \underline{\underline{A}}}_{\delta A_I} =$$

$$= \frac{E}{1+\nu} (\delta\vec{\alpha}_x \circ \vec{e}_x + \delta\vec{\alpha}_y \circ \vec{e}_y + \delta\vec{\alpha}_z \circ \vec{e}_z) \cdot \cdot (\delta\vec{\alpha}_x \circ \vec{e}_x + \delta\vec{\alpha}_y \circ \vec{e}_y + \delta\vec{\alpha}_z \circ \vec{e}_z) +$$

$$+ \frac{E}{1+\nu} \frac{\nu}{1-2\nu} (\delta A_I)^2 =$$

$$= \frac{E}{1+\nu} (\delta\vec{\alpha}_x \cdot \delta\vec{\alpha}_x + \delta\vec{\alpha}_y \cdot \delta\vec{\alpha}_y + \delta\vec{\alpha}_z \cdot \delta\vec{\alpha}_z) + \frac{E}{1+\nu} \frac{\nu}{1-2\nu} (\delta A_I)^2$$

Potenciális energia minimuma elv

Bizonyítás

Ha

$$E \geqq 0 \qquad \text{és} \qquad 0 < \nu < 0{,}5$$

akkor

$$\delta^2 \Pi_p = \frac{1}{2} \int\limits_{(V)} \delta \underline{\underline{F}} \cdot \cdot \delta \underline{\underline{A}} \, dV \geqq 0$$

$$\Pi_p \left[\vec{u}^* \right] \geqq \Pi_p \left[\vec{u} \right]$$

Bizonyítás

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$$E \geqq 0 \qquad \text{és} \qquad 0 < \nu < 0{,}5$$

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$$\delta^2 \Pi_p = \frac{1}{2} \int\limits_{(V)} \delta \underline{\underline{F}} \cdot \cdot \delta \underline{\underline{A}} \, dV \geqq 0$$

$$\Pi_p \left[\vec{u}^* \right] \geqq \Pi_p \left[\vec{u} \right]$$

Lagrange-féle variációs elv

Tétel

A teljes potenciális energiának az egzakt megoldás esetében szélső értéke (minimuma) van.

- Szükséges feltétel:
$$\delta\Pi_p = 0$$
- Elégséges feltétel:
$$\delta^2\Pi_p > 0$$

Lagrange-féle variációs elv

$$\delta\Pi_p = \int\limits_{(V)} \underline{\underline{F}} \cdot\cdot \delta \underline{\underline{A}} dV - \int\limits_{(A_p)} \delta\vec{u} \cdot \vec{p}_0 \, dA - \int\limits_{(V)} \delta\vec{u} \cdot \vec{f} \, dV =$$

$$= \int\limits_{(V)} \underline{\underline{F}} \cdot\cdot \delta \underline{\underline{D}} dV - \int\limits_{(A_p)} \delta\vec{u} \cdot \vec{p}_0 \, dA - \int\limits_{(V)} \delta\vec{u} \cdot \vec{f} \, dV =$$

$$= \int\limits_{(V)} \underline{\underline{F}} \cdot\cdot \left(\overset{\downarrow}{\delta\vec{u}} \circ \nabla \right) dV - \int\limits_{(A_p)} \delta\vec{u} \cdot \vec{p}_0 \, dA - \int\limits_{(V)} \delta\vec{u} \cdot \vec{f} \, dV =$$

$$= \int\limits_{(V)} \overset{\downarrow}{\delta\vec{u}} \cdot \underline{\underline{F}} \cdot \nabla dV - \int\limits_{(A_p)} \delta\vec{u} \cdot \vec{p}_0 \, dA - \int\limits_{(V)} \delta\vec{u} \cdot \vec{f} \, dV =$$

Lagrange-féle variációs elv

$$\delta\Pi_p = \int\limits_{(V)} \underline{\underline{F}} \cdot\cdot \delta \underline{\underline{A}} dV - \int\limits_{(A_p)} \delta\vec{u}\cdot\vec{p}_0 \, dA - \int\limits_{(V)} \delta\vec{u}\cdot\vec{f}\, dV =$$

$$= \int\limits_{(V)} \underline{\underline{F}} \cdot\cdot \delta \underline{\underline{D}} dV - \int\limits_{(A_p)} \delta\vec{u}\cdot\vec{p}_0 \, dA - \int\limits_{(V)} \delta\vec{u}\cdot\vec{f}\, dV =$$

$$= \int\limits_{(V)} \underline{\underline{F}} \cdot\cdot \left(\overset{\downarrow}{\delta\vec{u}} \circ \nabla\right) dV - \int\limits_{(A_p)} \delta\vec{u}\cdot\vec{p}_0 \, dA - \int\limits_{(V)} \delta\vec{u}\cdot\vec{f}\, dV =$$

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Lagrange-féle variációs elv

$$\delta\Pi_p = \int\limits_{(V)} \underline{\underline{F}} \cdot\cdot \delta\underline{\underline{A}} dV - \int\limits_{(A_p)} \delta\vec{u}\cdot\vec{p}_0\, dA - \int\limits_{(V)} \delta\vec{u}\cdot\vec{f}\, dV =$$

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Lagrange-féle variációs elv

$$\delta\Pi_p = \int\limits_{(V)} \underline{\underline{F}} \cdot\cdot \delta\underline{\underline{A}} dV - \int\limits_{(A_p)} \delta\vec{u}\cdot\vec{p}_0\, dA - \int\limits_{(V)} \delta\vec{u}\cdot\vec{f}\, dV =$$

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Lagrange-féle variációs elv

$$= \int\limits_{(V)} \overset{\downarrow}{\overline{\delta\vec{u} \cdot \underline{\underline{F}}}} \cdot \nabla dV - \int\limits_{(V)} \delta\vec{u} \cdot \overset{\downarrow}{\underline{\underline{F}}} \cdot \nabla dV - \int\limits_{(A_p)} \delta\vec{u} \cdot \vec{p}_0 \, dA - \int\limits_{(V)} \delta\vec{u} \cdot \vec{f} \, dV =$$

$$= \int\limits_{(A_p)} \delta\vec{u} \cdot \underline{\underline{F}} \cdot \vec{n} dA - \int\limits_{(V)} \delta\vec{u} \cdot \overset{\downarrow}{\underline{\underline{F}}} \cdot \nabla dV - \int\limits_{(A_p)} \delta\vec{u} \cdot \vec{p}_0 \, dA - \int\limits_{(V)} \delta\vec{u} \cdot \vec{f} \, dV =$$

$$= \int\limits_{(A_p)} \delta\vec{u} \cdot \left(\underline{\underline{F}} \cdot \vec{n} - \vec{p}_0\right) dA - \int\limits_{(V)} \delta\vec{u} \cdot \left(\underline{\underline{F}} \cdot \nabla + \vec{f}\right) dV = 0$$

Lagrange-féle variációs elv

$$= \int_{(V)} \overline{\delta\vec{u} \cdot \overset{\downarrow}{\underline{\underline{F}}}} \cdot \nabla dV - \int_{(V)} \delta\vec{u} \cdot \overset{\downarrow}{\underline{\underline{F}}} \cdot \nabla dV - \int_{(A_p)} \delta\vec{u} \cdot \vec{p}_0 \, dA - \int_{(V)} \delta\vec{u} \cdot \vec{f} \, dV =$$

$$= \int_{(A_p)} \delta\vec{u} \cdot \underline{\underline{F}} \cdot \vec{n} dA - \int_{(V)} \delta\vec{u} \cdot \overset{\downarrow}{\underline{\underline{F}}} \cdot \nabla dV - \int_{(A_p)} \delta\vec{u} \cdot \vec{p}_0 \, dA - \int_{(V)} \delta\vec{u} \cdot \vec{f} \, dV =$$

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Lagrange-féle variációs elv

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$$= \int\limits_{(A_p)} \delta\vec{u}\cdot\underline{\underline{F}}\cdot\vec{n} dA - \int\limits_{(V)} \delta\vec{u}\cdot\overset{\downarrow}{\underline{\underline{F}}}\cdot\nabla dV - \int\limits_{(A_p)} \delta\vec{u}\cdot\vec{p}_0\, dA - \int\limits_{(V)} \delta\vec{u}\cdot\vec{f}\, dV =$$

$$= \int\limits_{(A_p)} \delta\vec{u}\cdot\left(\underline{\underline{F}}\cdot\vec{n} - \vec{p}_0\right) dA - \int\limits_{(V)} \delta\vec{u}\cdot\left(\underline{\underline{F}}\cdot\nabla + \vec{f}\right) dV = 0$$

Lagrange-féle variációs elv

Ha V és A_p tetszőlegesek:

$$\underline{\underline{F}} \cdot \nabla + \vec{f} = \vec{0} \qquad \text{és} \qquad \underline{\underline{F}} \cdot \vec{n} = \vec{p}_0$$

Ha V és A_p rögzített (pl. egy konkrét feladatnál), akkor az egyensúlyi egyenlet és a dinamikai peremfeltétel csak *integrál értelem*ben teljesül.

Lagrange-féle variációs elv

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